Beyond Perturbation Theory in Inflation

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Outline

Introduction

- Inflation: $ds^2 = -dt^2 + a^2(t)d\boldsymbol{x}^2$ with $\ddot{a} > 0$

$$S = \int d^4x\sqrt{-g}\left[\frac{M_{\rm Pl}^2}{2}R + X - V(\phi) + \dots\right]$$

$X = -(\partial\phi)^2/2$. The background eq. for $\phi_0(t)$ is

$$\ddot{\phi}_0 + 3H\dot{\phi}_0 + V'(\phi_0) + \dots = 0$$

- de-Sitter space: Flat slicing

$$ds^2 = \frac{1}{H^2\eta^2}(-d\eta^2 + d\boldsymbol{x}^2)$$

Introduction

- Quantum Fluctuation ζ, ζ-gauge

$$\delta\phi = 0\,, \quad h_{ij} = a^2(t)\left[e^{2\zeta}\delta_{ij} + \gamma_{ij}\right]$$

- Free action of ζ

$$S = \int d\eta d^3x \frac{1}{2\eta^2 P_\zeta}\left[\zeta'^2 - (\partial_i\zeta)^2\right]$$

where $P_\zeta \equiv H^2/(2\epsilon M_{\rm Pl}^2)$

- Quantization as usual: $\zeta_{\boldsymbol{k}}(\eta) \sim \zeta^{\rm cl}_{\boldsymbol{k}}(\eta) a^\dagger_{\boldsymbol{k}} + \zeta^{\rm cl}_{\boldsymbol{k}}(\eta)^* a_{-\boldsymbol{k}}$

$\Rightarrow$ Scale invariant power spectrum

$$\langle \zeta_{\boldsymbol{k}} \zeta_{\boldsymbol{k}'} \rangle' = \frac{P_\zeta}{k^3}\,, \qquad P_\zeta \sim 10^{-8}$$

Introduction

- Interacting Hamiltonian: In-In formalism (weakly coupling limit)

$$\langle Q(\eta)\rangle = \langle 0|\,\bar{T} e^{i\int_{-\infty(1-i\epsilon)}^{\eta} H^I_{int}(\eta')d\eta'}\, Q^I(\eta)\, T e^{-i\int_{-\infty(1+i\epsilon)}^{\eta} H^I_{int}(\eta'')d\eta''}\,|0\rangle$$

$i\epsilon$-prescription $\Rightarrow$ Bunch-Davies vacuum $|0\rangle$

Introduction

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$i\epsilon$-prescription $\Rightarrow$ Bunch-Davies vacuum $|0\rangle$

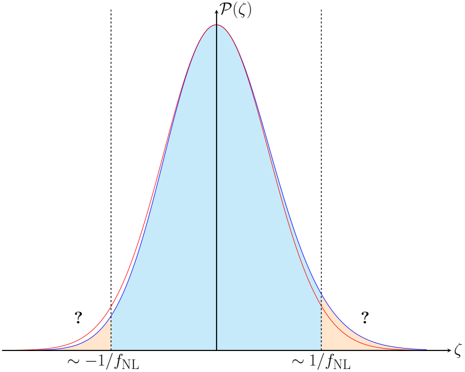
- Example: $\mathcal{L} = \frac{1}{2\eta^2 P_\zeta}\left[\zeta'^2 - (\partial_i\zeta)^2\right] + \frac{\lambda}{4! P_\zeta^2}\zeta'^4$,

$$\frac{\langle\zeta\zeta\zeta\rangle}{P_\zeta^{3/2}} \sim f_{NL} P_\zeta^{1/2} \ll 1\ , \qquad \frac{\langle\zeta\zeta\zeta\zeta\rangle}{P_\zeta^{2}} \sim g_{NL} P_\zeta \sim \lambda \ll 1$$

The expansion parameter is just λ

- Tails of the distribution

$$\mathcal{P}_{\rm PT}[\zeta] \sim \exp\left\{ -\frac{\zeta^2}{2P_\zeta} - \frac{\langle\zeta\zeta\zeta\rangle}{2P_\zeta^3}\zeta^3 - \frac{\langle\zeta\zeta\zeta\zeta\rangle}{2P_\zeta^4}\zeta^4 \dots \right\}$$
$$= \exp\left\{ -\frac{\zeta^2}{2P_\zeta}\left[1 + \frac{\langle\zeta\zeta\zeta\rangle}{P_\zeta^2}\zeta + \frac{\langle\zeta\zeta\zeta\zeta\rangle}{P_\zeta^3}\zeta^2 + \dots\right]\right\}$$



$$\frac{\langle\zeta\zeta\zeta\rangle}{P_\zeta^2}\zeta \sim f_{NL}\zeta,$$
$$\frac{\langle\zeta\zeta\zeta\zeta\rangle}{P_\zeta^3}\zeta^2 \sim g_{NL}\zeta^2$$

The expansion parameter now depends on size of ζ

Main Idea

Unlikely events at the tails

$\Vert$

Semi-classical limit ($\hbar \to 0$)

The wavefunction of the Universe (WFU) $\sim e^{iS/\hbar}$ will be computed semi-classically

Introduction

- Primordial black hole formation: occurs around horizon re-entry

The mass fraction of PBH is

$$\beta(M) = \int_{\zeta_c}^{\infty} \mathcal{P}[\hat{\zeta}] d\hat{\zeta} \,, \qquad \hat{\zeta}(\boldsymbol{x}) = \int \frac{d^3 k}{(2\pi)^3} W(k) \zeta(\boldsymbol{k}) e^{i\boldsymbol{k}\cdot\boldsymbol{x}}$$

$\Rightarrow$ The formation is sensitive to $\zeta \sim 1$

$\Rightarrow$ The pert. theory is still valid for $f_{NL}\zeta \sim f_{NL} \ll 1$

(Single field slow-roll: $f_{NL} \sim \mathcal{O}(\epsilon, \eta)$, K-Inflation: $f_{NL} \sim (1 - 1/c_s^2)$, $|f_{NL}^{equil}| < 80$)

$\Rightarrow$ To study PBH formation one needs to go beyond perturbation theory

Analogy in QM: Compute the wavefunction in the Semi-classical limit

Euclidean Path integral

- From Path integral to the wavefunction of the ground state

$$\langle x_f | e^{-H(\tau_f - \tau_i)/\hbar} | x_i \rangle = \sum_n e^{-E_n(\tau_f - \tau_i)/\hbar} \Psi_n(x_f) \Psi_n^*(x_i)$$

$$\Psi_0(x_f)\Psi_0^*(x_i) e^{-E_0 T/\hbar} = \lim_{T \to \infty} \int_{x(\tau_i)=x_i}^{x(\tau_f)=x_f} \mathcal{D}x(\tau) e^{-S_E[x(\tau)]/\hbar}$$

- Expand $x(\tau) = x_{cl}(\tau) + y(\tau)$

$$\Psi_0(x_f) = N \; e^{-S_E[x_{cl}]/\hbar} \int_{y(\tau_i)=0}^{y(\tau_f)=0} \mathcal{D}y(\tau) e^{-\frac{1}{\hbar}\left(\frac{1}{2}\frac{\delta^2 S}{\delta x^2} y^2 + \frac{1}{3!}\frac{\delta^3 S}{\delta x^3} y^3 + \ldots\right)}$$

$$\Psi_0(x_f) \simeq \; \mathcal{I}[x_f] \; e^{-S_E[x_{cl}]/\hbar}$$

Anharmonic oscillator

- $V(x) = \hbar\omega\left[\frac{1}{2}(\frac{x}{d})^2 + \lambda(\frac{x}{d})^4\right]$, $\quad d \equiv \sqrt{\hbar/m\omega}$

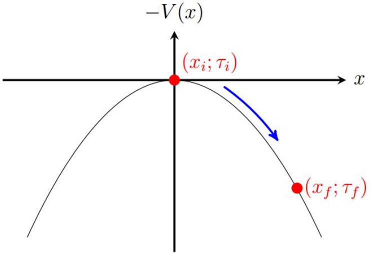
$\Rightarrow$ The PT breaks down when $\lambda x_f^2/d^2 \equiv \bar{x}^2/2 \sim 1$

- In Euclidean space,
$\mathcal{L} = \frac{1}{2}m\dot{x}^2 + V(x)$

- Real path connecting $x(\tau_i) = x_i$ and $x(\tau_f) = x_f$

- For $T = \tau_f - \tau_i \to \infty \Rightarrow E = 0$

$\Rightarrow$ The real path with infinite amount of times is the one with zero energy



$$x(\tau) = -\frac{d}{\sqrt{2\lambda}\sinh(\omega\tau)}$$

Anharmonic oscillator: Scaling argument

- Recall the wavefunction $\Psi_0(x_f) \simeq \mathcal{I}[x_f] e^{-S_E[x_{\rm cl}]/\hbar}$
- The Euclidean action is

$$S_E[x(\tau)] = \int_{\tau_{\rm i}}^{\tau_{\rm f}} d\tau \left\{ \frac{1}{2} m\dot{x}^2 + \hbar\omega \left[\frac{1}{2} \left(\frac{x}{d} \right)^2 + \lambda \left(\frac{x}{d} \right)^4 \right] \right\}$$

Rescaling $x \to (\sqrt{\hbar/\lambda})x$, then

$$\frac{S_E[x_{\rm cl}(\tau)]}{\hbar} \sim \frac{1}{\lambda} F(\lambda x_f^2/d^2)$$

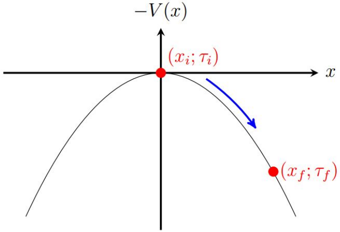
- The prefactor of $\mathcal{I}[x_f]$ goes as $\lambda^0 G(\lambda x_f^2/d^2) \Leftrightarrow$ 1-loop diagrams
- Neglect the higher-order in $\lambda \Leftrightarrow$ higher-loop diagrams

Anharmonic oscillator: Ground-state wavefunction

- The on-shell action with zero energy

$$\frac{S_E[x_{cl}(\tau)]}{\hbar} = \frac{1}{\hbar}\int_{\tau_i}^{\tau_f} d\tau \; m\dot{x}^2 = \frac{1}{6\lambda}\left[(1+\bar{x}^2)^{3/2} - 1\right]$$

$$\mathcal{I}(x_f) = \mathcal{N}\sqrt{\frac{m}{2\pi i\hbar v_i v_f \int_0^{x_f} \frac{\mathrm{d}x'}{v^3(x')}}}, \qquad \bar{x}^2 \equiv 2\lambda x_f^2/d^2$$



$$x(\tau) = -\frac{d}{\sqrt{2\lambda}\sinh(\omega\tau)}$$

$$\Psi_0(x_f) = \mathcal{N}\frac{\exp\left\{-\frac{1}{6\lambda}\left[\left(1+\bar{x}^2\right)^{3/2} - 1\right]\right\}}{(1+\bar{x}^2)^{1/4}(1+\sqrt{1+\bar{x}^2})^{1/2}}\left(1+\mathcal{O}(\lambda)f(\bar{x})\right)$$

This is valid for arbitrary $\bar{x}$.

Free theory

- The wavefunction of the Universe $\Psi[\zeta_0(\boldsymbol{x})] = \int_{BD}^{\zeta_0(\boldsymbol{x})} \mathcal{D}\zeta \; e^{iS[\zeta]/\hbar}$
- The free saddle point is (Maldacena 02)

$$\zeta^{cl}_{\boldsymbol{k}}(\eta) = \zeta^0_{\boldsymbol{k}} \frac{(1-ik\eta)e^{ik\eta}}{(1-ik\eta_f)e^{ik\eta_f}}$$

$\Rightarrow$ $i\epsilon$-prescrip. selects the correct BC at early times

$$iS[\zeta_{cl}] = i\int_{\boldsymbol{k}} \frac{1}{2P_\zeta \eta_f^2} \zeta^{cl}_{-\boldsymbol{k}} \partial_\eta \zeta^{cl}_{\boldsymbol{k}} \Big|_{\eta=\eta_f} = \int_{\boldsymbol{k}} \frac{1}{2P_\zeta} \left(\frac{ik^2}{\eta_f} - k^3 + \dots \right) \zeta^0_{-\boldsymbol{k}} \zeta^0_{\boldsymbol{k}}$$

$\int_{\boldsymbol{k}} = \int d^3k/(2\pi)^3$

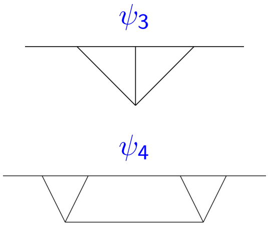
$\Rightarrow$ The WFU is a Gaussian distribution

Interacting theory

- In perturbation theory, the WFU can be expanded as

$$\Psi = \exp\left[\frac{1}{2}\int d^3x d^3y \langle \mathcal{O}(\boldsymbol{x})\mathcal{O}(\boldsymbol{y})\rangle \zeta(\boldsymbol{x})\zeta(\boldsymbol{y}) + \frac{1}{6}\int d^3x d^3y d^3z \langle \mathcal{O}(\boldsymbol{x})\mathcal{O}(\boldsymbol{y})\mathcal{O}(\boldsymbol{z})\rangle \zeta(\boldsymbol{x})\zeta(\boldsymbol{y})\zeta(\boldsymbol{z}) + \dots\right]$$

$\Rightarrow$ The on-shell action amounts to computing tree-level Witten diagrams



Cosmological correlators:

$$\langle \zeta_{\boldsymbol{k}}\zeta_{-\boldsymbol{k}}\rangle' = \frac{-1}{2Re\langle \mathcal{O}_{\boldsymbol{k}}\mathcal{O}_{-\boldsymbol{k}}\rangle'}$$

$$\langle \zeta_{\boldsymbol{k}_1}\zeta_{\boldsymbol{k}_2}\zeta_{\boldsymbol{k}_3}\rangle' = \frac{2Re\langle \mathcal{O}_{\boldsymbol{k}_1}\mathcal{O}_{\boldsymbol{k}_2}\mathcal{O}_{\boldsymbol{k}_3}\rangle'}{\Pi_i(-2Re\langle \mathcal{O}_{\boldsymbol{k}_i}\mathcal{O}_{-\boldsymbol{k}_i}\rangle')}$$

Non-linear WFU

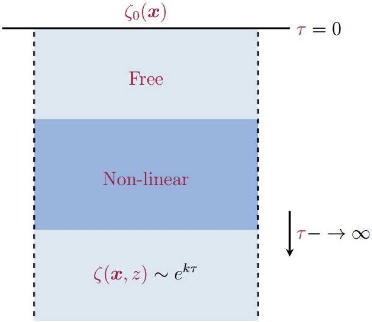
- Boundary conditions: BD at early times and ζ_0 at late times
- Find the non-linear classical solution to the EoM
- Compute the WFU in the semi-classical limit

$$\Psi[\zeta_0(\boldsymbol{x})] \sim e^{iS[\zeta_{cl}]/\hbar}$$

- The derivative coupling e.g. ζ'^4

$\Rightarrow$ Deep inside horizon: free theory

$\Rightarrow$ Outside horizon: non-linear term is switched off



EFT of Inflation

• The single coupling ζ'^4 can be justified in EFT of Inflation for large quartic operator (Senatore & Zaldarriaga 11)

$$\mathcal{L}_{EFT} = -M_{\rm Pl}^2 \dot{H}\left(\dot{\pi}^2 - \frac{(\partial_i \pi)^2}{a^2}\right) + M_4^4(16\dot{\pi}^4 - 32\dot{\pi}^3(\partial_\mu \pi)^2 + \ldots)$$

The coeff. of cubic operators can be set to zero: $M_2^4(\delta g^{00})^2$, $M_3^4(\delta g^{00})^3$

• $\pi \to \pi_c$, $\mathcal{O}(\pi_c^{N>4})$ are suppressed by $g_{\rm NL} \sim M_4^4/(|\dot{H}|M_{\rm Pl}^2) \lesssim 10^6$

$$iS = i\int d^3x d\eta \left\{\frac{1}{2\eta^2 P_\zeta}\left[\zeta'^2 - (\partial_i \zeta)^2\right] + \frac{\lambda}{4! P_\zeta^2}\zeta'^4\right\}, \quad \zeta = -H\pi_c/\dot{\phi}_0$$

The Euclidean EoM reads

$$-\zeta'' + \frac{2}{\tau}\zeta' - \partial_i^2 \zeta - \frac{\lambda}{2P_\zeta}\tau^2 \zeta'^2 \zeta'' = 0$$

Scaling argument of $S[\zeta_{\rm cl}]$

- Recall the WFU: $\Psi[\zeta_0(\boldsymbol{x})] \sim e^{-S_E[\zeta_{\rm cl}]}$
- The Euclidean action ($\eta = -i\tau$)

$$S_E \equiv -\int d^3x d\tau \left\{ \frac{1}{2\tau^2 P_\zeta} \left[\zeta'^2 + (\partial_i \zeta)^2 \right] + \frac{\lambda}{4! P_\zeta^2} \zeta'^4 \right\}$$

- Rescaling $\zeta \to \zeta/\sqrt{\lambda}$, then

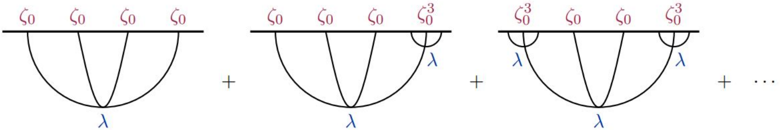
$$S_E[\zeta_{\rm cl}] = \frac{1}{\lambda} F\left(\lambda \zeta_0^2 / P_\zeta\right)$$

- The relevant expansion parameter is $\lambda \zeta_0^2 / P_\zeta$
- Neglect the prefactor, $\lambda^0 G(\lambda \zeta_0^2 / P_\zeta)$, and the higher orders in λ

Witten diagrams

- Tree-level graphs, captured by semiclassical method:

$$\frac{F(\lambda\zeta_0^2)}{\lambda} \sim \zeta_0^2 + \lambda\zeta_0^4 + \lambda^2\zeta_0^6 + \lambda^3\zeta_0^8 + \ldots$$



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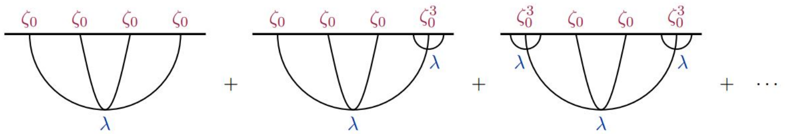
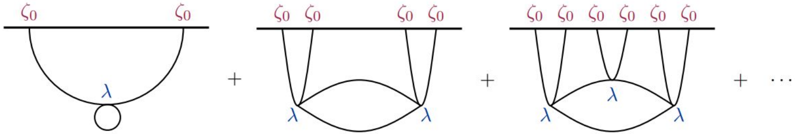


- 1-loop graphs, would be captured by the prefactor:

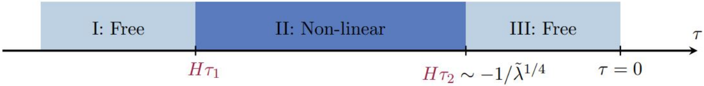
$$\lambda^0 G(\lambda\zeta_0^2) \sim \lambda\zeta_0^2 + \lambda^2\zeta_0^4 + \lambda^3\zeta_0^6 + \dots$$



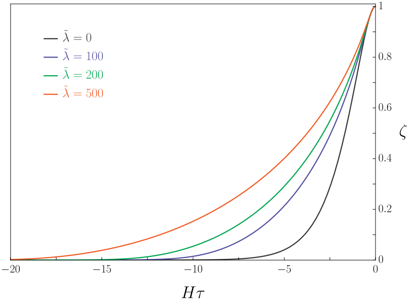
Approximation using ODE

- The derivative coupling only affects the modes of similar wavelength

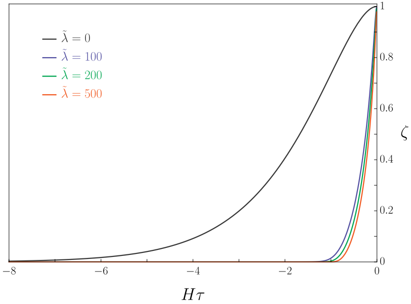
$$-\zeta'' + \frac{2}{\eta}\zeta' + H^2\zeta - \frac{\lambda}{2P_\zeta}\eta^2\zeta'^2\zeta'' = 0\,, \quad \tilde{\lambda} = \lambda\zeta_0^2/P_\zeta$$



$\zeta \to \zeta_0\zeta$



$\tau \to \tilde{\lambda}^{1/2}\tau$



Approximation using ODE

- The rescaling in τ gives the behaviour for large λ

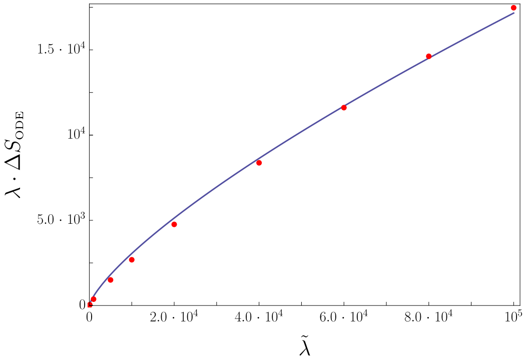
$$\Delta S_{ODE} = -\frac{\zeta_0^2}{P_\zeta}\int_{\tau_\mathrm{i}}^{\tau_\mathrm{f}} d\tau \left\{ \frac{1}{2\tau^2}\left[\zeta'^2 + H^2(\zeta^2 - 1)\right] + \frac{\tilde{\lambda}}{4!}\zeta'^4 \right\} = \frac{1}{\lambda}F(\tilde{\lambda})$$

$\tilde{\lambda} = \lambda\zeta_0^2/P_\zeta$

$$\Delta S_{ODE} \sim \frac{1}{\lambda}\tilde{\lambda}^{3/4}$$

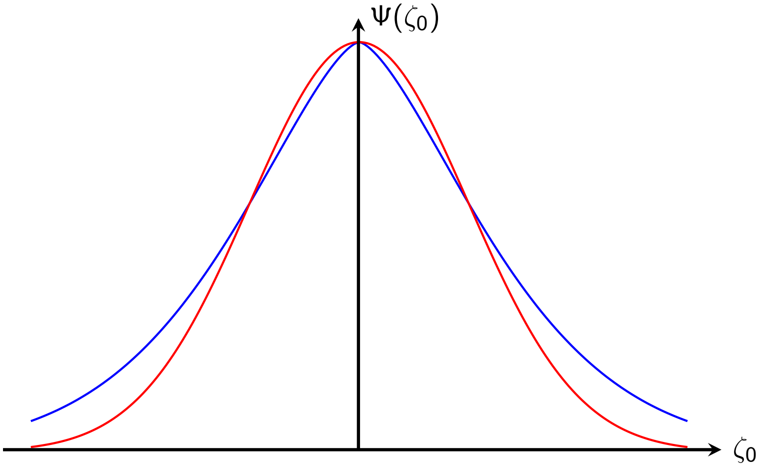
$$\Downarrow$$

$$\Psi[\zeta_0] \sim \exp\left[-\frac{\zeta_0^{3/2}}{\lambda^{1/4}P_\zeta^{3/4}}\right]$$



Approximation using ODE

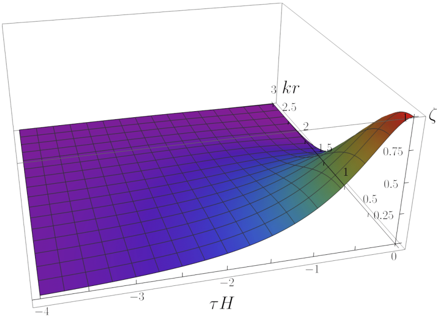
- $\Psi_G \sim \exp(-\zeta_0^2/2)$, $\Psi \sim \exp(-\zeta_0^{3/2}/2)$: Ψ is heavier than Ψ_G



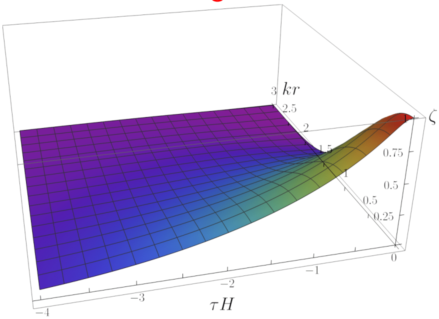
PDE with Gaussian profile

- The Gaussian profile at η_c: $\zeta(r) \sim \zeta_0 e^{-r^2}$

small λ



large λ

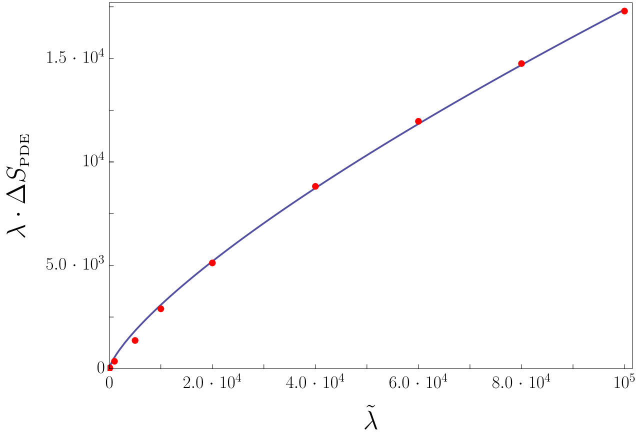


- For small λ, it reduces to perturbative result

PDE with Gaussian profile

- The on-shell action

$$\Delta S_{PDE} \sim \frac{1}{\lambda}\tilde{\lambda}^{3/4} \Rightarrow \Psi \sim \exp(-\zeta_0^{3/2}/\lambda^{1/4})$$



Future Directions: Inflation

- Explore PBH formation $\zeta_c \sim 1$, $\mathcal{P}[\zeta^0_{\boldsymbol{k}}] = |\Psi[\zeta_0]|^2$

$$\mathcal{P}[\zeta_c] = \mathcal{N}^{-1} \int \mathcal{D}[\zeta^0_{\boldsymbol{k}}]\ \mathcal{P}[\zeta^0_{\boldsymbol{k}}]\ \Theta(\hat{\zeta}[\zeta^0_{\boldsymbol{k}}] - \zeta_c)$$

$\hat{\zeta}[\zeta^0_{\boldsymbol{k}}] = \int_{\boldsymbol{k}} W(k) \zeta^0_{\boldsymbol{k}} e^{i\boldsymbol{k}\cdot\boldsymbol{x}}$

- Generalize to
  - Different interactions
  - Slow-roll inflation
  - Tensor mode γ_{ij}
- Connection to large number of legs limit (e.g. Badel et al. 20)
- Any implication for AdS/CFT ? Compute the exact Z for given source ?

Two fields in dS

• Idea: take one field to be on the background and compute the other non-perturbatively

$$S = \int d\eta d^3x \left[\frac{1}{2\eta^2 H^2}(\sigma'^2 - (\partial_i \sigma)^2) + \frac{1}{2\eta^2 H^2}(\chi'^2 - (\partial_i \chi)^2) - \frac{\lambda}{\eta^4 H^3}\chi\sigma^2 \right]$$

• For $k_\chi \ll k_\sigma$ we have

$$S_\sigma = \int d\eta d^3x \left[\frac{1}{2\eta^2 H^2}(\sigma'^2 - (\partial_i \sigma)^2) - \frac{\alpha H^2}{2\eta^4}\sigma^2 \right]$$

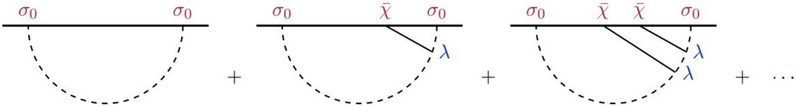
where $\alpha \equiv 2\lambda\bar{\chi}/H$. This is just a massive scalar field on dS whose power spectrum at late times is

$$\langle \sigma_{\boldsymbol{k}} \sigma_{-\boldsymbol{k}} \rangle' \simeq \frac{H^2}{2k^{3-\frac{2}{3}\alpha}} = \frac{H^2}{2k^{3-\frac{4}{3}\lambda\bar{\chi}/H}}$$

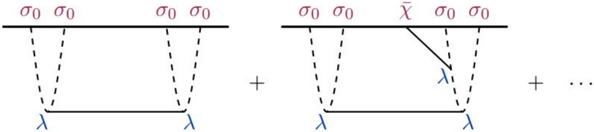
We have resummed all powers in $\lambda\bar{\chi}$

Two fields in dS

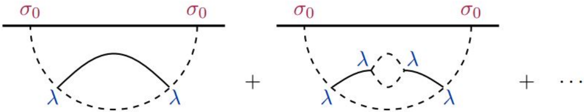
- Tree-level diagrams, enhanced by $\bar{\chi}$ and resummed



- Tree-level exchange diagrams, with fewer powers of $\bar{\chi}$



- Loop diagrams, subleading in λ



Work in progress: Spatial derivative coupling

- The spatial derivative interaction $(\partial_i\zeta)^4$

$$S = \int d^3x d\eta \frac{1}{P_\zeta}\left\{\frac{1}{2\eta^2}\left[\zeta'^2 - (\partial_i\zeta)^2\right] \pm \frac{\lambda}{4!}(\partial_i\zeta)^4\right\}$$

- All possible subtleties:

- The $+$ sign $\Rightarrow$ Gradient inst.

- The $-$ sign (healthy) $\Rightarrow$ the solution becomes complex for large λ

  - Study QM in p-space for $\hat{x}^2 + \hat{x}^4$, $\hat{x} \sim d/dp$, $\Psi(p) \sim e^{i\sigma(p)/\hbar}$

$$\frac{p^2}{2m} + V(-\sigma'(p)) = E \ , \quad \sigma(p_f) = \int^{p_f} dp \ V^{-1}\left(E - \frac{p^2}{2m}\right)$$

  - There are complex saddle points depending on p_f

Work in progress: Two fields model

- Two field model: $S = \int d\eta d^3x \left[\mathcal{L}_\sigma^0 + \mathcal{L}_\chi^0 - \frac{\lambda}{\Lambda^4}(\partial_i\sigma)^2(\partial_i\chi)^2\right]$
- Treat χ as a background for σ ($k_\chi \ll k_\sigma$)

$$\sigma_{\boldsymbol{k}}'' - \frac{2}{\eta}\sigma_{\boldsymbol{k}}' + (1+\alpha\eta^2)k^2\sigma_{\boldsymbol{k}} = 0, \quad \alpha \equiv \frac{2\lambda(\partial_i\bar{\chi})^2H^2}{\Lambda^4}$$

- The power spectrum of σ is

$$\langle\sigma_{\boldsymbol{k}}\sigma_{-\boldsymbol{k}}\rangle' = \frac{\pi}{8k^{3/2}\alpha^{3/4}} \frac{e^{-\pi k/(4\sqrt{\alpha})}}{\left|\Gamma\left(\frac{5}{4}+\frac{ik}{4\sqrt{\alpha}}\right)\right|^2}$$

$\Rightarrow$ Not analytic around $\alpha = 0$

Work in progress: Two fields model

- The result matches with PT for small α

$$\langle \sigma_{\boldsymbol{k}} \sigma_{-\boldsymbol{k}} \rangle' = \frac{H^2}{2k^3} \left(1 - \frac{5\lambda(\partial_i \bar{\chi})^2 H^2}{2\Lambda^4 k^2} \right)$$

- For large α

$$\langle \sigma_{\boldsymbol{k}} \sigma_{-\boldsymbol{k}} \rangle' \simeq \frac{H^2}{k^{3/2} \alpha^{3/4}}$$

- The Wavefunction of the Universe is

$$\Psi[\sigma_0] \sim \exp[-\alpha^{3/4} \sigma_0^2]$$

$\Rightarrow$ This is the WFU of σ in the large background of χ

Backup Inflation

EFT of Inflation: Large field limit

• Take $M_2 = M_3 = 0$. The operators $\pi^{N>4}$ are suppressed by $g_{\rm NL}$. $\pi \to -\pi$ is an approx. symmetry when $g_{\rm NL} >> 1$. The operators with odd power will then be suppressed by $g_{\rm NL}$.

• Loop corrections to $M_2(\delta g^{00})^2$ and $M_3(\delta g^{00})^3$ also are suppressed by $g_{\rm NL}$ since their leading terms are odd in π.

• What about $(\delta g^{00})^n$?

- For n odd, these will be suppressed by approx. symmetry

- For n even, no suppression $\Rightarrow$ consider all of them or the loop integral can be cut at $\Lambda < \Lambda_U$. At least they are down by $(\Lambda/\Lambda_U)^{\#}$. Otherwise UV completion is needed

• $\delta g^{00} = 1 + g^{00} \to -2\dot{\pi} + (\partial_\mu \pi)^2$ under $t \to t + \pi$

EFT of Inflation: Large field limit

- $\mathcal{L}_\zeta$ from $(\delta g^{00})^4$: $(\zeta = -H\pi)$

$$S_\zeta = \int d^4x\sqrt{-g}\,\frac{|\dot{H}|M_{\rm Pl}^2}{H^2}\left[(\partial_\mu\zeta)^2 + g_{\rm NL}\frac{1}{H^2}\dot{\zeta}^4 + g_{\rm NL}\frac{1}{H^3}\dot{\zeta}^3(\partial_\mu\zeta)^2 + \ldots\right]$$

- Comparison with $\mathcal{L}_2$:

$$\frac{\mathcal{L}_4}{\mathcal{L}_2} \sim g_{\rm NL}\zeta^2 \sim 1\,, \quad \frac{\mathcal{L}_5}{\mathcal{L}_2} \sim g_{\rm NL}\zeta^3 = g_{\rm NL}\zeta^2\zeta \ll 1$$

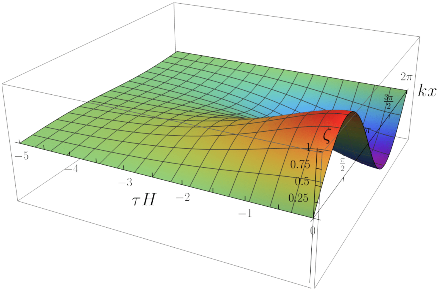
for $g_{\rm NL} \gg 1$ (Exp. $g_{\rm NL} \ll 10^6$).

- $\mathcal{L}_5$ becomes important $(g_{\rm NL}\zeta^3 \sim 1)$ when $g_{\rm NL}\zeta^2 \gtrsim g_{\rm NL}^{1/3}$
- If $\zeta \sim 1 \Rightarrow$ all the terms inside each $(\delta g^{00})^n$ are important, e.g. $\mathcal{L}_4/\mathcal{L}_5 \sim \zeta$.

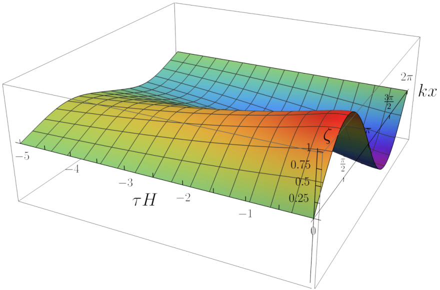
PDE with sinusoidal profile

- The Gaussian profile at η_c: $\zeta(r) \sim \zeta_0 \sin(kx)$

small λ



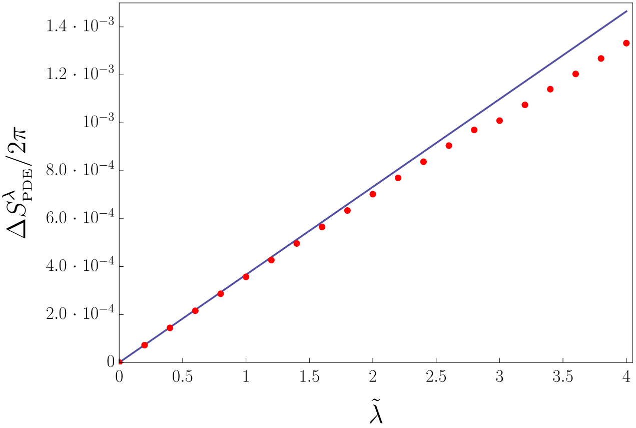
large λ



- This can be easily checked with perturbation theory

Perturbative check with PDE sine wave

- Perturbative 4-pt: $iS'_{\rm int} = 3\zeta_0^2\tilde{\lambda}k^3/(8192P_\zeta)$
- Numerical: $\Delta S^{\tilde{\lambda}}_{PDE} = -(\Delta S_{PDE} - \Delta S^0_{PDE})$



Non-linearity $\neq$ Breaking down of EFT

• It is not generally true that when the non-linearities become important the EFT we are considering necessarily breaks down

• Take GR in which all the non-linear terms are controlled by diff-invariance but the EFT (GR) is still valid as long as ∂/Λ is small

• It's the same spirit as one considers the Vainshtein mechanism where there is a regime which is dominated by non-linear term, but the EFT is still valid

• The issue of instabilities has to be taken care of separately. We are not saying that all the solutions to the non-linear EoM are healthy (also it depends on the background we are expanding around). The presence of instabilities might signal the need of the UV completion.

Non-linearity $\neq$ Breaking down of EFT

• Take $X + X^2$ in which the UV completion is known, but it does not mean that once the non-linearity becomes important the IR theory breaks down

• One can also take the DBI action and work out all the non-linear terms of DBI around $\phi_0(t)$. Again the EFT action is valid even though the non-linearities become important

• Of course the question whether the UV completion exists or not is interesting on its own, but it does not really mean that the EFT breaks down once the non-linear terms are important

Non-linearity $\neq$ Breaking down of EFT

- UV completion of $X + X^2$
- $\mathcal{L}_{\rm IR} = \mathcal{P}(X)$ with constant X background $\Rightarrow$ Ghost + gradient inst.

The non-linear terms are contained in X^2

- $\mathcal{L}_{UV} = -|\partial\phi|^2 - \lambda(|\phi|^2 - v^2)^2$ $\phi = \phi_0 e^{i\pi}$, $\langle\phi_0\rangle = v^2 - \frac{X}{2\lambda}$, $X = -(\partial\pi)^2$
- Around $\phi_0(t)$, $X + \beta X^2$ yields

$$S_E = i\int d\eta d^3x \frac{1}{P_\zeta}\left\{\frac{1}{2\eta^2}[\zeta'^2 + (\partial_i\zeta)^2] + \frac{\lambda}{4!}(\partial_i\zeta)^4 + \frac{\lambda c_s^2}{6\eta}\zeta'(\partial_i\zeta)^2 + \frac{\lambda c_s^2}{12}\zeta'^2(\partial_i\zeta)^2 + \frac{\lambda c_s^4}{6\eta}\zeta'^3 + \frac{\lambda c_s^4}{4!}\zeta'^4\right\}$$

No suppression due to small c_s^2 since $c_s^2 = (1+\beta\dot{\phi}_0^2)/(1+3\beta\dot{\phi}_0^2) \in (1/3, 1)$

- The suppression happens for $-X + \beta X^2$ for small $c_s^2 \in (0, 1/3)$

Analytic Continuation

• The Euclidean rotation $\eta \to iz$ can be easily shown in perturbation theory - order by order of the solution given the source is analytic

$$\zeta(\eta, \boldsymbol{k}) = K(\eta, \boldsymbol{k})\zeta^0_{\boldsymbol{k}} + \int_{-\infty(1-i\epsilon)}^{\eta_c} \mathrm{d}\eta' \, G(\eta, \eta'; \boldsymbol{k}) \frac{\delta S_{int}}{\delta\zeta(\eta', \boldsymbol{k})}$$

$K(\eta, \boldsymbol{k})$ is bulk-boundary propagator

$$K(\eta, \boldsymbol{k}) = \frac{(1 - ik\eta)}{(1 - ik\eta_c)} e^{ik(\eta - \eta_c)}$$

Analytic Continuation

The bulk-bulk propagator

$$G(\eta,\eta';\boldsymbol{k}) = \frac{-iH^2}{2k^3}\left[\phi_+(\eta)\phi_-(\eta') - \frac{\phi_-(\eta_c)}{\phi_+(\eta_c)}\phi_+(\eta')\phi_+(\eta)\right], |\eta| > |\eta'|$$
$$= \frac{-iH^2}{2k^3}\left[\phi_+(\eta')\phi_-(\eta) - \frac{\phi_-(\eta_c)}{\phi_+(\eta_c)}\phi_+(\eta')\phi_+(\eta)\right]\ , |\eta| < |\eta'|$$

$$\phi_-(\eta) \equiv (1+ik\eta)e^{-ik\eta}\ , \quad \phi_+(\eta) \equiv (1-ik\eta)e^{ik\eta}$$

Analytic Continuation beyond PT

Euclidean Path-Integral

• Is this the only real solution in Euclidean space ?

- If yes, the Picard-Lefschetz thimbles $\Rightarrow$ it is the only saddle that contributes to path integral

- If not, there are contributions from complex saddles and one needs to worry about the Stokes phenomenon (the jump in asymptotic behaviour $\Rightarrow$ other saddles can dominate)

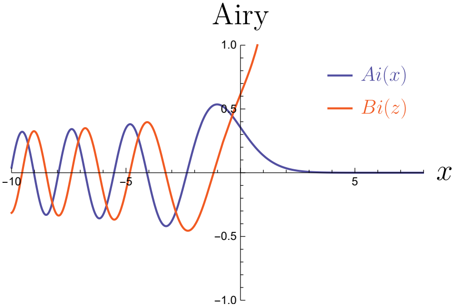
• In QM with quartic potential, there is only one real solution (Serone, Spada, and Villadoro 17)

Stokes phenomenon of Airy function

$$Ai(x) \equiv \frac{1}{2\pi} \int_{-\infty}^{\infty} dt \ e^{\frac{i}{3}t^3 + ixt}$$

- For $x \in \mathcal{R}^+$, two imaginary saddles: $\pm i\sqrt{|x|} \Rightarrow$ Oscillatory
- For $x \in \mathcal{R}^-$, two real saddles: $\pm\sqrt{|x|} \Rightarrow$ Decaying and growing (neglect the growing behaviour)

- Changing from negative to positive the integral is dominated by different saddles (Stokes phenomenon)



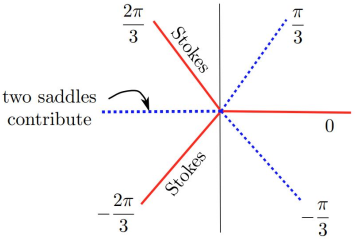
Stokes phenomenon of Airy function

- For complex Airy function

$$Ai(z) \sim z^{-1/4} \exp\left(-\frac{2}{3} z^{2/3}\right), \quad Bi(z) \sim z^{-1/4} \exp\left(\frac{2}{3} z^{2/3}\right)$$

- Stokes lines: $Im(z^{2/3}) = 0 \Rightarrow \arg(z) = 0, \pm 2\pi/3$
- Anti-Stokes lines: $Re(z^{2/3}) = 0 \Rightarrow \arg(z) = \pm\pi/3, \pi$

- $Ai(z)$ is subdominant in $-\pi/3 < \arg(z) < \pi/3$, dominant otherwise
- $Bi(z)$ is dominant in $-\pi/3 < \arg(z) < \pi/3$, subdominant otherwise



(Mariño, Pasquetti, and Putrov 10)

Analytic Continuation beyond PT

Real time Path-Integral

- Not well-defined because of huge oscillatory behaviour
- Need to give $i\epsilon$ to have a well-defined integral
  - How many saddle points are there ? All of them contribute to path-integral ?
  - Are they analytic ? If yes, the full rotation to Euclidean can be done
- For real time instanton, the on-shell action with $i\epsilon$ is the same as the on-shell Euclidean action (Cherman and Unsal 14)
- For real time quantum tunneling, the solution with $i\epsilon$ admits poles and zeros in complex t-plane (Turok 14)

Analytic Continuation beyond PT

• From dS to Euclidean AdS

$$ds^2 = \frac{1}{H^2\eta^2}(-d\eta^2 + d\boldsymbol{x}^2)$$

Perform $\eta \to iz$ and $H \to i/L$

$$ds^2 = \frac{L^2}{z^2}(dz^2 + d\boldsymbol{x}^2)$$

• It has been shown that the functional integral can be analytically continued from dS to EAdS once restricted on the analytic functions (Harlow and Stanford 11)